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Highly irregular digraphs

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Abstract

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A connected digraph D is *highly irregular* if the vertices of *out-neighborhood* of each vertex $v \in V(D)$ have different *out-degrees*. In this paper, we investigate some problems concerning the existence of highly irregular digraphs with special properties, with particular focus on highly irregular directed trees as well as their independence numbers.

1. Introduction

Alavi et al. [1] introduced the class of highly irregular graphs and established some properties of a highly irregular graph. We now extend that definition to digraphs and consider highly irregular digraphs which are opoosite, in a certain sense, to regular digraphs. Some similar results are obtained. For a digraph D, $v \in V(D)$, we call

$$N_O(v) = \{u \mid u \in V(D), (v, u) \in A(D)\},$$

$$N_I(v) = \{u \mid u \in V(D), (u, v) \in A(D)\},$$

$$d_O(v) = |N_O(v)| \quad \text{and} \quad d_I(v) = |N_I(v)|,$$

out-neighborhood, in-neighborhood, out-degree and in-degree, of v, respectively.

If $N_O(v) = \{u_1, u_2, \ldots, u_t\}$ and $d_O(u_i) = d_i$ $(1 \leqslant i \leqslant t)$, then we call $(d_1, d_2, \ldots, d_t)$ the out-degree sequence of v. A ditree is an orientation graph of a tree such that there exists exactly one vertex (called root) with in-degree 0 and other vertices have in-degree 1.

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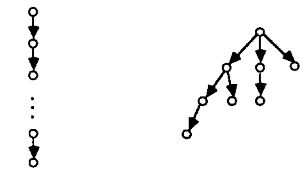


a highly irregular dipath a highly irregular ditree

Fig. 1.

Definition 1.1. Let D be a digraph. If $\forall v \in V(D)$, $d_0(u) \neq d_0(w)$ holds for any $u, w \in N_0(v)$, then D is called a highly irregular digraph.

For example, a directed path is a highly irregular digraph; see Fig. 1.

Fact 1.2. *Any directed path is a highly irregular digraph.*

Fact 1.3. *Any ditree obtained from an orientation of a highly irregular tree is a highly irregular ditree* (*see Fig.* 1).

Fact 1.4. *Any highly irregular graph, as a digraph, obtained by replacing each edge by two opposite arcs, is a highly irregular digraph. So, the family of highly irregular graphs is a subfamily of highly irregular digraphs.*

Fact 1.5. *For any positive integer n, there exists one highly irregular digraph with order n.*

A directed path on n vertices or the digraph obtained from an orientation of a complete graph with n vertices such that its out-degree sequence $(n-1, n-2, \ldots, 1, 0)$ is a highly irregular digraph with order n.

Fact 1.6. *For any positive integer d, there exists a highly irregular digraph with maximum out-degree d.*

This can be seen by considering an orientation of a highly irregular tree with maximum degree d such that the root has out-degree d, or the digraph obtained from an orientation of K_{d+1} such that its out-degree sequence is $(d, d-1, d-2, \ldots, 1, 0)$.

Fact 1.7. *For any positive integers d and n, with $1 \leqslant d < n$, there exists a connected highly irregular digraph with order n and maximum out-degree d.*

Proof. Let G_1 be an orientation of K_{d+1} such that its out-degree sequence is $(d, d-1, d-2, \ldots, 1, 0)$, and let $v_{d+2}, \ldots, v_n$ be other vertices. Let D be a digraph with $V(D) = V(G_1) \cup \{v_{d+2}, \ldots, v_n\}$ and $A(D) = A(G_1) \cup \{(v_i, v) \mid d+2 \leqslant i \leqslant n\}$, where $v \in V(G_1)$; then D is a highly irregular digraph with the maximum out-degree being d. $\square$

2. Highly irregular ditrees

Similar to Fact 1.6, for any nonnegative integer d, there exists a highly irregular ditree with maximum out-degree d. If we construct ditrees T_k as follows:

$$T_0=\{v_0\},$$

$$T_1=\{(v_1,v_0)\},$$

$$T_2=T_1\cup T_0\cup\{v_2\}\cup\{(v_2,w_i)\mid w_i \text{ is the root of } T_i, i=0,1\}$$

$$\vdots$$

$$T_k=T_0\cup T_1\cup\cdots\cup T_{k-1}\cup\{v_k\}\cup\{(v_k,w_i)\mid w_i \text{ is the root of } T_i, i=0,1,2,\ldots,k-1\},$$

then it is easy to see that T_k is a highly irregular ditree with maximum out-degree k and $|V(T_k)|=2^k$. Next, we show that 2^k is the smallest possible order for such a ditree.

Theorem 2.1. *The order of a highly irregular ditree with maximum out-degree d is at least* 2^d, *and* T_d *is the only such ditree of order* 2^d.

Proof. By induction on d, when $d=1$, it is easy to see that $|V(T)|\geqslant 2$ for any highly irregular ditree T with maximum out-degree 1 and only T_1 has order 2. Assume that the theorem is true for all highly irregular ditrees with maximum out-degree $d<k$. Now let T be a highly irregular ditree with maximum out-degree k such that $|V(T)|$ is as small as possible, then we want to prove $T\cong T_k$. It is easy to see that there exists exactly one vertex with maximum out-degree k, which is the root of T. Otherwise, we can obtain a highly irregular ditree T' such that the maximum out-degree of T' is k and $|V(T')|<|V(T)|$, contradicting the choice of T. Let v be the root of T, $N_0(v)=\{v_1,v_2,\ldots,v_k\}$. Let $T_1',T_2',\ldots,T_k'$ be the components of $T-v$ containing $v_1,v_2,\ldots,v_k$, respectively, and $d_0(v_i)=n_i$ for $1\leqslant i\leqslant k$. Without loss of generality, assume $n_1<n_2<\cdots<n_k$. Then $n_1=0, n_2=1,\ldots,n_k=k-1$. Hence, $n_k<k$. Furthermore, $T_i'\cong T_{i-1}$ for $1\leqslant i\leqslant k$; otherwise, we can replace T_i' by T_{i-1} to get a highly irregular ditree with maximum out-degree k and with order $<|V(T)|$, contradicting the choice of T. So, $T\cong T_k$ and $|V(T)|=|V(T_k)|=2^k$. By induction principle, the theorem follows. □

Theorem 2.2. *For any positive integers d and n, with* $n\geqslant 2^d$, *there exists a highly irregular ditree with order n and maximum out-degree d.*

Proof. Let d and n be any positive integers such that $n\geqslant 2^d$. Let $u\in V(T_d)$, with $d_0(u)=0$, and let P_{n-2^d+1} be a dipath from v to w. Then we obtain a new digraph G by identifying u and v. It is easy to see that G is a highly irregular ditree of order n; so, the theorem follows. □

In [1], Alavi et al. discussed the independence number β of highly irregular trees and found an upper bound $\beta(T) \leqslant 9n/14$ for such a tree T. Now we derive a sharp bound for highly irregular ditrees.

Theorem 2.3. *If T is a highly irregular ditree of order $n \geqslant 2$, then $\beta(T) \leqslant \lfloor \frac{2}{3}n \rfloor$. Furthermore, when $n = 3m+2$, there exists a highly irregular ditree T^*_{3m+2} such that $\beta(T^*_{3m+2}) = \lfloor \frac{2}{3}(3m+2) \rfloor$.*

Proof. Let T be an arbitrary highly irregular ditree with root v, X be an independent set of T with $\beta(T)$ vertices and $Y = V(T) - X$. Furthermore, let $e(X, Y)$ be the number of arcs from X to Y. Choose $y_1 \in Y$ so that y_1 has the maximum number of out-neighbors in X; denote by $N_0(y_1)$ the set of out-neighbors of y_1 in X, and let $n(y_1) = |N_0(y_1)|$. Next choose $y_2 \in Y$ so that y_2 has the maximum number of out-neighbors in $X - N_0(y_1)$; denote by $N_0(y_2)$ the set of out-neighbors of y_2 in $X - N_0(y_1)$, and let $n(y_2) = |N_0(y_2)|$. Continuing in this manner, we produce a sequence $y_1, y_2, \ldots, y_k$ such that the out-neighborhood of Y in X is contained in $\bigcup_{i=1}^k N_0(y_i)$. Now we complete the proof by discussing the following two cases.

(a) The root v is not in X. Then $\sum_{i=1}^k n(y_i) = |X|$.

Since T is highly irregular, the sum of the out-degrees and the in-degrees of $n(y_i)$ vertices of $N_0(y_i)$ is at least

$$\sum_{j=1}^{n(y_i)} j = \binom{n(y_i)+1}{2}.$$

It, therefore, follows that

$$e(X, Y) + e(Y, X) \geqslant \sum_{i=1}^k \binom{n(y_i)+1}{2}.$$

In this case, we will prove that $|X| < \frac{2}{3}n$. Suppose now, to the contrary, that $|X| \geqslant \frac{2}{3}n$; then $|X| \geqslant 2|Y|$.

If $n(y_1) = 1$, then $n(y_1) = n(y_2) = \cdots = n(y_k) = 1$ and $|X| = \sum_{i=1}^k n(y_i) = k \leqslant |Y|$, contradicting to $|X| \geqslant 2|Y|$. Thus, we assume that $n(y_i) \geqslant 2$ for $1 \leqslant i \leqslant s$.

Let $w = \sum_{i=1}^s [n(y_i) - 2]$. Then $\sum_{i=1}^s n(y_i) = 2s + w$ and

$$|X| = \sum_{i=1}^s n(y_i) + \sum_{i>s} n(y_i) \leqslant 2s + w + |Y| - s = s + w + |Y|,$$

i.e., $s \geqslant |X| - (|Y| + w)$.

We now claim that

$$\sum_{i=1}^s \binom{n(y_i)+1}{2} \geqslant 3s + 3w.$$

In fact,

$$\sum_{i=1}^{s}\binom{n(y_i)+1}{2}=\sum_{i=1}^{s}\frac{n(y_i)(n(y_i)+1)}{2}=\frac{3}{2}\sum_{i=1}^{s}n(y_i)+\sum_{i=1}^{s}\frac{n(y_i)(n(y_i)-2)}{2}$$

$$\geqslant\frac{3}{2}(2s+w)+\sum_{i=1}^{s}\frac{3}{2}[n(y_i)-2]=3s+\frac{3}{2}w+\frac{3}{2}w=3s+3w.$$

Since T is a ditree, we have

$$\begin{aligned}
n&>e(X,Y)+e(Y,X)\\
&\geqslant\sum_{i=1}^{k}\binom{n(y_i)+1}{2}=\sum_{i=1}^{s}\binom{n(y_i)+1}{2}+\sum_{i>s}\binom{n(y_i)+1}{2}\\
&\geqslant 3s+3w+|X|-(2s+w)\\
&=s+2w+|X|\\
&\geqslant |X|-(|Y|+w)+2w+|X|\\
&=3|X|+w-(|X|+|Y|)\\
&=3|X|+w-n.
\end{aligned}$$

It follows that $2n>3|X|+w\geqslant 3|X|$. Therefore, $|X|<\frac{2}{3}n$, contradicting our assumption. Hence, $|X|<\frac{2}{3}n$, i.e., $|X|\leqslant\lfloor\frac{2}{3}n\rfloor$.

(b) The root v is in X.

Let $u_1,u_2,\ldots,u_m$ be the vertices of T such that $(v,u_i)\in A(T)$. Then $T-v$ consists of m ditrees T_i with root u_i $(1\leqslant i\leqslant m)$. Let $X_i=X\cap V(T_i)$. Since $v\in X$, $u_i\notin X$, i.e., $u_i\notin X_i$, $1\leqslant i\leqslant m$. By (a), we know $|X_i|<\frac{2}{3}|V(T_i)|$ for $1\leqslant i\leqslant m$. So, $|X|<\frac{2}{3}(n-1)+1$. This implies that $|X|\leqslant\lfloor\frac{2}{3}n\rfloor$.

Let T^*_{3m+2} be the ditree shown in Fig. 2. It is easy to see that T^*_{3m+2} is a highly irregular ditree and $\beta(T^*_{3m+2})=2m+1$ (In fact, $X=\{u_i\,|\,1\leqslant i\leqslant m+1\}\cup\{w_i\,|\,1\leqslant i\leqslant m\}$ is an independent set and $\beta(T^*_{3m+2})=|X|=2m+1=\lfloor\frac{2}{3}(3m+2)\rfloor$). $\square$

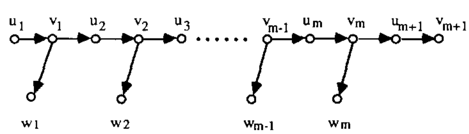
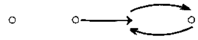


Fig. 2.

3. Highly irregular digraphs containing a given digraph as an induced subdigraph

In [1], authors present a result that may be considered an analogue of König's theorem for highly irregular graphs. Now we give a similar result for highly irregular digraphs.

Theorem 3.1. *Every digraph of order $n \geqslant 2$ is an induced subdigraph of a highly irregular digraph of order $4n-4$.*

Proof. Let D be a digraph of order $n \geqslant 2$. If $n=2$, then $D \cong D_i$ ($i=1$ or 2 or 3), where D_i are the digraphs shown in Fig. 3. Let $T' =$ . Then D_1, D_2, D_3 are the induced subdigraphs T'. Thus, we assume $n \geqslant 3$. Let D' be another copy of D, where $V(D) = \{v_i \mid 1 \leqslant i \leqslant n\}$, $V(D') = \{v'_i \mid 1 \leqslant i \leqslant n\}$, and v'_i corresponds to $v_i (1 \leqslant i \leqslant n)$. We construct a new digraph H as follows

$$V(H) = V(D) \cup V(D') \cup \{u_i \mid 1 \leqslant i \leqslant n-2\} \cup \{u'_i \mid 1 \leqslant i \leqslant n-2\},$$

$$A(H) = A(D) \cup A(D') \cup E',$$

where E' contains the following arcs:

(a) For $v_i, v_j \in V(D)$, if $(v_i, v_j) \notin A(D)$, $(v_j, v_i) \in A(D)$, then $(v_i, v'_j), (v'_i, v_j) \in E'$.

(b) For $v_i, v_j \in V(D)$, if $(v_i, v_j) \notin A(D), (v_j, v_i) \notin A(D)$, then $(v_i, v'_j), (v_j, v'_i), (v'_i, v_j), (v'_j, v_i) \in E'$.

(c) $(v_i, v'_i), (v'_i, v_i) \in E'$, $1 \leqslant i \leqslant n-1$.

(d) For $1 \leqslant j \leqslant n-2$, (u_j, v_i), $(v_i, u_j) \in E'$, where $1 \leqslant i \leqslant j$.

(e) For $1 \leqslant j \leqslant n-2$, (u'_j, v'_i), $(v'_i, u'_j) \in E' (1 \leqslant i \leqslant j)$.

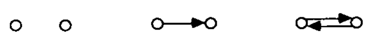
It is easy to see that H contains D as an induced subdigraph. Moreover, for $1 \leqslant i \leqslant n$,

$$d_0^H(v_i) = d_0^H(v'_i) = 2n-1-i,$$

whereas, for $1 \leqslant i \leqslant n-2$,

$$\deg_0^H(u_i) = \deg_0^H(u'_i) = i.$$

So, H has exactly two vertices of out-degree i for each i, $1 \leqslant i \leqslant 2n-2$. Therefore, H is a highly irregular digraph. This completes the proof of Theorem 3.1. □



D_1 D_2 D_3

Fig. 3.

Next, for each digraph in a special class of digraphs, we present a highly irregular digraph H such that H contains that digraph as its induced subdigraph and $|V(H)|=2n-1$.

Theorem 3.2. *If D is a digraph such that $n-1=d_0(v_1)=d_0(v_2)=\cdots=d_0(v_n)$, where $V(D)=\{v_i \mid 1\leqslant i\leqslant n\}$, then there exists a highly irregular digraph H with order $2n-1$ such that D is an induced subdigraph of H.*

Proof. For $n=1$, it is trivial. So, we assume $n\geqslant 2$. Let Q be an orientation graph of K_{n-1} and $V(Q)=\{u_i \mid 1\leqslant i\leqslant n-1\}$ with $d_0(u_i)=i-1$. We construct a new digraph H as follows:

$$V(H)=V(D)\cup V(Q),$$

$$A(H)=A(D)\cup A(Q)\cup E,$$

where E is constituted by the following arcs:

For $1\leqslant j\leqslant n-1$, $(v_i, u_j)\in E$, $1\leqslant i\leqslant j$.

Then H is a highly irregular digraph containing D as its induced digraph. □

4. Highly irregular digraphs with preassigned groups

Alavi and Ruiz [2], discussed the highly irregular graph with preassigned groups. Note that any highly irregular graph, as a digraph, is a highly irregular digraph; so, the same result as in [2] is true for highly irregular digraphs. Now we prove that it is true for highly irregular asymmetric digraphs.

Theorem 4.1. *Every finite group is isomorphic to the automorphism group of a highly irregular asymmetric digraph.*

Proof. If the group is trivial, then the trivial digraph satisfies the theorem; so, we assume that the group is nontrivial.

Let Γ be the given group with Cayley color digraph $D=D_\Delta(\Gamma)$, where $\Delta=\{h_1, h_2, \ldots, h_d\}$ generates Γ and $e\notin\Gamma$. We can obtain a highly irregular digraph H with automorphism group isomorphic to Γ from Cayley color digraph as follows.

For each vertex x of D and for each color $h_i\in\Delta$, we replace the arc (x, xh_i) by the digraph indicated in Fig. 4, where T_{2i} is the highly irregular ditree defined in Section 2 and u_x^i is the root of T_{2i} and T'_{2i+1} is the following highly irregular ditree.

$$T'_{2i+1}\cong\{v\}\cup T_0\cup T_1\cup\cdots\cup T_{d-1}\cup T_{d+1}\cdots\cup T_{2i+1}$$
$$\cup\{(v, w_j) \mid w_j \text{ are the roots of } T_j, 0\leqslant j\leqslant 2i+1, j\neq d\}.$$

and v_x^i is the root of T'_{2i+1}.

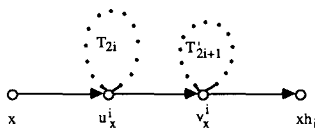


Fig. 4.

From the structure we know that each vertex x of D has the out-degree sequence $(2, 4, 6, \ldots, 2d)$, the out-degree sequence of u_x^i is $(0, 1, 2, \ldots, 2i-1, 2i+2)$ and the out-degree sequence of v_x^i is $(0, 1, \ldots, d-1, d, d+1, \ldots, 2i+1)$. So, H is a highly irregular digraph and the automorphism group of H is isomorphic to Γ. Furthermore, H is asymmetric. Therefore, the theorem is true. □

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